

STRONG NONNEGATIVITY AND SUMS OF SQUARES ON REAL VARIETIES

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ABSTRACT. Motivated by scheme theory, we introduce strong nonnegativity on real varieties, which has the property that a sum of squares is strongly nonnegative. We show that this algebraic property is equivalent to nonnegativity for nonsingular real varieties. Moreover, for singular varieties, we reprove and generalize obstructions of Gouveia and Netzer to the convergence of the theta body hierarchy of convex bodies approximating the convex hull of a real variety.

1. INTRODUCTION

The relationship between nonnegative polynomials and sums of squares of polynomials on real varieties is a classical subject, dating back to Hilbert. In real algebraic geometry, a large body of research is dedicated to understanding the gap between these families. At the same time, this subject has recently become important in the emerging field of convex algebraic geometry, where it is relevant to the effectiveness of computing convex hulls of algebraic varieties. This in turn has been intimately related to the geometry of feasible regions of semidefinite programs (see [6] and references therein). Motivated by this and inspired by scheme theory, we introduce an intermediate class of polynomials which we call *strongly nonnegative*. This class is particularly useful for understanding the role that singularities on real varieties play in obstructing sums of squares representations.

We begin by exploring the basic properties of strong nonnegativity, showing in particular in Theorem 2.10 that strong nonnegativity at a point implies nonnegativity in a neighborhood of that point, and that the converse holds for nonsingular points. In the singular case, we study obstructions to the theta body hierarchy [5] of convex bodies approximating the convex hull of a real variety. The strength of this approximation is governed by the sums of squares representability of linear functions on a variety. We are able to recover very transparently in Theorem 4.4 the obstructions produced by Gouveia and Netzer in [4] to convergence of this hierarchy. The same argument gives us Corollary 4.3, a generalized version of their obstruction. Finally, Proposition 5.2 shows that our construction behaves well in the context of the foundational constructions of Gouveia, Parrilo and Thomas in [5].

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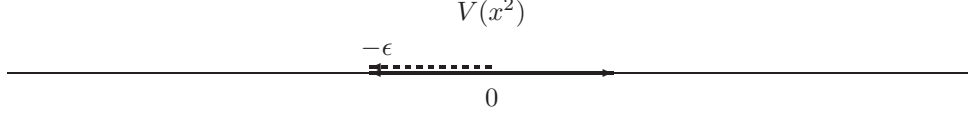


FIGURE 1. Motivating example

2. STRONG NONNEGATIVITY

Our convention throughout, given an ideal $I \subseteq \mathbb{R}[x_1, \dots, x_n]$, is to use $V_{\mathbb{R}}(I)$ for the real vanishing set of I , and use $V(I)$ in relation to concepts depending on the ring $\mathbb{R}[x_1, \dots, x_n]/I$, which we will denote by A . Formally, $V(I)$ is the closed subscheme $\text{Spec}(A) \subseteq \mathbb{R}^n$, but our definitions will be in terms of A , so no knowledge of schemes is required. All of our ring homomorphisms are assumed to be \mathbb{R} -algebra homomorphisms.

We begin by introducing our stricter definition of nonnegativity. Our motivating example is the following:

Example 2.1. Suppose $I \subseteq \mathbb{R}[x]$ is the ideal generated by x^2 . Then set-theoretically, we have $V_{\mathbb{R}}(I)$ equal to the origin. Thus, the function x is nonnegative on $V_{\mathbb{R}}(I)$. However, one easily checks that x is not a sum of squares modulo I .

From a more scheme-theoretic perspective, we should think of $V(I)$ as not consisting only of the origin, but also including an infinitesimal thickening in both directions – in particular, in the negative direction. Thus, we should not think of x as being nonnegative on the *scheme* $V(I)$.

Recall that if $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ is an ideal, then the points of $V_{\mathbb{R}}(I)$ correspond precisely to (\mathbb{R} -algebra) homomorphisms $A \rightarrow \mathbb{R}$, where $A = \mathbb{R}[x_1, \dots, x_n]/I$. The homomorphism obtained from a given $P \in V_{\mathbb{R}}(I)$ is simply given by evaluating polynomials at P . Thus, one may rephrase nonnegativity as saying that f is nonnegative if its image under any homomorphism $A \rightarrow \mathbb{R}$ is nonnegative. Our definition will consider a broader collection of such homomorphisms. In particular, given a point of $V_{\mathbb{R}}(I)$ corresponding to $\varphi : A \rightarrow \mathbb{R}$, it is standard that the (scheme-theoretic) tangent space to $V(I)$ at the point is in bijection with homomorphisms $A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^2)$ which recover φ after composing with the unique homomorphism $\mathbb{R}[\epsilon]/(\epsilon^2) \rightarrow \mathbb{R}$, which necessarily sends ϵ to 0.

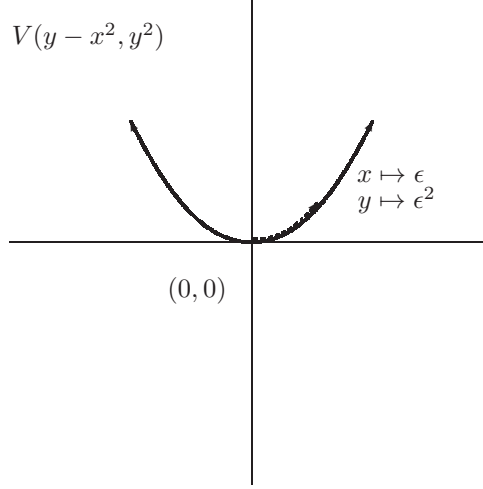
In Example 2.1, a tangent vector in the “negative direction” is given by the homomorphism $\mathbb{R}[x]/(x^2) \rightarrow \mathbb{R}[\epsilon]/(\epsilon^2)$ sending x to $-\epsilon$. If we consider $-\epsilon$ to be “negative”, we may thus consider the function x to take a negative value on this tangent vector to $V(I)$. We formalize and generalize this idea by considering also higher-order infinitesimal arcs, as follows.

Definition 2.2. Given $f \in \mathbb{R}[\epsilon]/(\epsilon^m)$, $f = a_0 + a_1\epsilon + \dots + a_{m-1}\epsilon^{m-1}$, we say f is **nonnegative** if $f = 0$, or $a_N > 0$ where $N = \min\{j : a_j \neq 0\}$.

Note that $\mathbb{R}[\epsilon]/(\epsilon^m)$ has a unique homomorphism to \mathbb{R} , necessarily sending ϵ to 0. We say that $\varphi : A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ is **at** P for (a necessarily unique) $P \in V_{\mathbb{R}}(I)$ if P is the point corresponding to the composed homomorphism $A \rightarrow \mathbb{R}$.

Definition 2.3. Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal, and $A := \mathbb{R}[x_1, \dots, x_n]/I$. Given $P \in V_{\mathbb{R}}(I)$, we say $f \in A$ is **strongly nonnegative** at P if for every $m \geq 0$ and for every \mathbb{R} -algebra homomorphism

$$\varphi : A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$$

FIGURE 2. $-y$ is not strongly nonnegative on $V(y - x^2, y^2)$

at P , we have $\varphi(f)$ is nonnegative. We say f is **strongly nonnegative** on $V(I)$ if it is strongly nonnegative at P for all $P \in V_{\mathbb{R}}(I)$.

We begin with some basic observations on the property of strong nonnegativity.

Proposition 2.4. *Given $f \in A$, we have the following statements.*

- (1) *If f is strongly nonnegative at $P \in V_{\mathbb{R}}(I)$, then f is nonnegative at P .*
- (2) *If f is strictly positive at $P \in V_{\mathbb{R}}(I)$, then f is strongly nonnegative at P .*
- (3) *If f is a sum of squares, then f is strongly nonnegative.*

Proof. We obtain (1) immediately by setting $m = 1$ in the definition, since this yields the evaluation map at P .

For (2), given any homomorphism $\varphi : A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ at P , by definition we have that composing with $\mathbb{R}[\epsilon]/(\epsilon^m) \rightarrow \mathbb{R}$ gives the evaluation map at P , under which f is strictly positive by hypothesis. But then if we write $\varphi(f) = a_0 + a_1\epsilon + \dots + a_{n-1}\epsilon^{n-1}$, we must have $a_0 = f(P) > 0$, and thus $\varphi(f)$ is nonnegative. Since φ was arbitrary at P , we conclude f is strongly nonnegative at P .

Finally, for (3) if $f = \sum_{i=1}^r h_i^2$, and $\varphi : A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ is an \mathbb{R} -algebra homomorphism, then the leading term of each $(\varphi(h_i))^2$ is nonnegative, and hence so is that of $\varphi(f)$. \square

We will show in Theorem 2.10 that in fact if f is strongly nonnegative at P , then it is nonnegative on a neighborhood of P , and that the converse holds if P is a nonsingular point of $V(I)$. Of course, the converse does not hold in general.

Example 2.5. Consider $I = (y - x^2, y^2) \subseteq \mathbb{R}[x, y]$, and $P = (0, 0)$ the only point of $V_{\mathbb{R}}(I)$. Then $-y$ is not strongly nonnegative on $V(I)$: under the homomorphism $\varphi : \mathbb{R}[x, y]/I \rightarrow \mathbb{R}[\epsilon]/(\epsilon^3)$ at P sending x to ϵ and y to ϵ^2 , we have $\varphi(-y) = -\epsilon^2$ is not nonnegative.

On the other hand, y is strongly nonnegative on $V(I)$ by Proposition 2.4 (3), since $y = x^2$ modulo I .

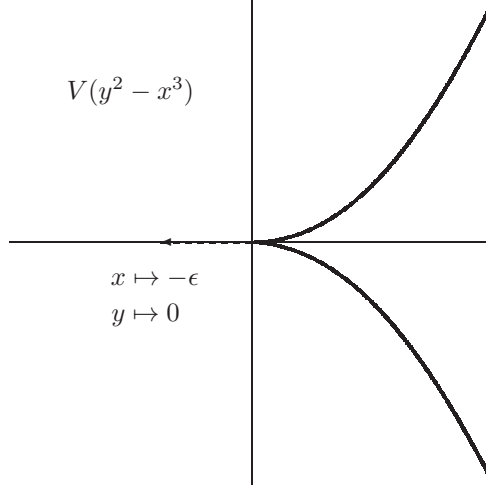


FIGURE 3. The negative direction $(-1, 0)$ at $(0, 0)$ on $V(y^2 - x^3)$

We also give an example where $V(I)$ is reduced (i.e., I is radical) for which strong nonnegativity is strictly stronger than nonnegativity.

Example 2.6. Consider the ideal $I = (y^2 - x^3) \subseteq \mathbb{R}[x, y]$, and the function $f(x, y) = x$ nonnegative on $V_{\mathbb{R}}(I)$. Note that $f(x, y)$ is negative on the direction $(-1, 0)$ at the singular point $(0, 0)$ of $V(I)$. This is realized algebraically by the homomorphism

$$\varphi : \mathbb{R}[x, y]/(y^2 - x^3) \rightarrow \mathbb{R}[\epsilon]/(\epsilon^2), \quad \varphi(x) = -\epsilon, \quad \varphi(y) = 0,$$

at P , which proves f is not strongly nonnegative since the leading coefficient of $\varphi(f) = \varphi(x) = -\epsilon$ is negative. Thus, f cannot be a sum of squares.

This example may be made compact by instead setting $I = (y^2 - x^3 + x^4)$.

Remark 2.7. A suitable local version of Proposition 2.4 (3) may be described in terms of the complete local ring \hat{A}_P of $V(I)$ at P . Specifically, if f is a sum of squares in \hat{A}_P , then f is strongly nonnegative at P . The proof is the same, since any homomorphism $A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ at P factors through the complete local ring.

Remark 2.8. Note that if there exists a homomorphism $A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ such that the image of f has its leading term in odd degree, then f is not strongly nonnegative, since we may change the sign of the coefficient by composing with the automorphism of $\mathbb{R}[\epsilon]/(\epsilon^m)$ sending ϵ to $-\epsilon$.

We now consider the deeper question of the relationship between strong nonnegativity at a point, and nonnegativity in a neighborhood. This requires concepts related to nonsingularity, which for the sake of clarity, we now recall:

Definition 2.9. Given $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ and $P \in V_{\mathbb{R}}(I)$, set $A = \mathbb{R}[x_1, \dots, x_n]/I$, and let $\mathfrak{m}_P \subseteq A$ be the maximal ideal of A consisting of polynomials vanishing at P . Note that because $P \in V_{\mathbb{R}}(I)$, we have $A/\mathfrak{m}_P \cong \mathbb{R}$. The **cotangent space** of $V(I)$ at P is the real vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$, and the **tangent space** of $V(I)$ at P is the dual space $\text{Hom}_{\mathbb{R}}(\mathfrak{m}_P/\mathfrak{m}_P^2, \mathbb{R})$. The **dimension** of $V(I)$ at P is the dimension of the local ring $A_{\mathfrak{m}_P}$. Finally, $V(I)$ is **nonsingular** at P if the tangent space at P has dimension equal to the dimension of $V(I)$ at P .

We then have the following:

Theorem 2.10. *Given $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ and a point $P \in V_{\mathbb{R}}(I)$, suppose that $f \in A := \mathbb{R}[x_1, \dots, x_n]/I$ is strongly nonnegative at P . Then f is nonnegative in a (real) neighborhood of P . Moreover, the converse holds if P is a nonsingular point of $V(I)$.*

It will be convenient to extend our terminology as follows:

Definition 2.11. Suppose $P \in V_{\mathbb{R}}(I)$. Then a homomorphism $\varphi : A \rightarrow \mathbb{R}[[t]]$ is **at P** if the preimage of the ideal generated by t is the (maximal) ideal of functions vanishing at P .

The following is the main technical lemma, which does not involve strong nonnegativity and which applies without the nonsingularity hypothesis. Although the result is well known in real algebraic geometry, we include it for the convenience of the reader.

Lemma 2.12. *Given $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ a point $P \in V_{\mathbb{R}}(I)$, and $f \in A := \mathbb{R}[x_1, \dots, x_n]/I$, the following are equivalent:*

- (1) *f is nonnegative in a (real) neighborhood of P ;*
- (2) *for every homomorphism $\varphi : A \rightarrow \mathbb{R}[[t]]$ at P taking values in locally convergent power series, we have that the leading term of $\varphi(f)$ is nonnegative.*
- (3) *for every homomorphism $\varphi : A \rightarrow \mathbb{R}[[t]]$ at P , we have that the leading term of $\varphi(f)$ is nonnegative.*

Geometrically, a homomorphism $A \rightarrow \mathbb{R}[[t]]$ at P taking values in locally convergent power series defines an *analytic arc* at P ; that is, we obtain an analytic map from $(-c, c) \subseteq \mathbb{R}$ to $V(I)$ for some $c > 0$, sending 0 to P . By analogy, we think of an arbitrary homomorphism $A \rightarrow \mathbb{R}[[t]]$ at P as a *formal arc* at P . Thus, the content of the lemma may be viewed as saying that f is nonnegative on a neighborhood of P if and only if it is nonnegative on every analytic arc at P , if and only if it is nonnegative on every formal arc at P .

Proof. We first show that (1) and (2) are equivalent. The implication that (1) implies (2) is straightforward. Indeed, if $\varphi(f)$ has negative leading term for some φ , then for t_0 sufficiently small and positive, we would have $\varphi(f)(t_0) < 0$, and because

$$\varphi(f)(t_0) = f(\varphi(x_1)(t_0), \dots, \varphi(x_n)(t_0)),$$

the points $(\varphi(x_1)(t_0), \dots, \varphi(x_n)(t_0))$ would yield points arbitrarily close to P with f negative. For the converse, we appeal to the Curve Selection Lemma (see Theorem VII.4.2 and Remarks VII.4.3 of [1]). Suppose that (1) is false. Then P is in the closure of the set $S = \{x \in V_{\mathbb{R}}(I) : f(x) < 0\}$. Now, S is semi-algebraic, so by the Curve Selection Lemma, there exists a half-branch at P of an algebraic curve contained in $V(I)$ such that away from P , the half-branch is contained in S . This half-branch is in particular analytic, so it is defined by a homomorphism $\varphi : A \rightarrow \mathbb{R}[[t]]$ at P taking values in locally convergent power series, and moreover we have that $\varphi(f)$ is negative for all sufficiently small positive values of t . We conclude that $\varphi(f)$ has negative leading coefficient, as desired.

We now move on to proving the equivalence of (2) and (3). Of course, (3) trivially implies (2). The key ingredient for the converse is an Artin-style approximation theorem. Suppose we have $\varphi : A \rightarrow \mathbb{R}[[t]]$ at P such that $\varphi(f)$ has negative leading

term. A theorem of Greenberg [7] (which is a special case of Artin's approximation theorem; see also §VII.3 of [1]) asserts that we can replace φ by a homomorphism φ' which takes values in locally convergent power series and agrees with φ to arbitrarily high order; that is, for any fixed N , we can find φ' such that for all $g \in A$, we have that the first N terms of $\varphi'(g)$ agree with the first N terms of $\varphi(g)$. In particular, we may choose φ' such that $\varphi'(f)$ still has negative leading term, and we thus conclude the desired result. \square

The proof of Theorem 2.10 is almost immediate from Lemma 2.12.

Proof of Theorem 2.10. First suppose that f is not nonnegative on any neighborhood of P . Then Lemma 2.12 implies that there exists a homomorphism $A \rightarrow \mathbb{R}[[t]]$ under which f has negative leading term. If the leading term occurs in degree $m-1$, truncating from $\mathbb{R}[[t]]$ to $\mathbb{R}[\epsilon]/(\epsilon^m)$ via $t \mapsto \epsilon$ then shows that f is not strongly nonnegative.

Conversely, suppose that f is nonnegative on a neighborhood of P in $V_{\mathbb{R}}(I)$, and $V(I)$ is nonsingular at P . Because nonsingularity is equivalent to smoothness in characteristic 0, by a generalization of Hensel's lemma if we have a homomorphism $\varphi : A \rightarrow \mathbb{R}[\epsilon]/(\epsilon^m)$ at P , we can lift to $\mathbb{R}[\epsilon]/(\epsilon^{m'})$ for m' arbitrarily large (see Proposition 2.2.15 and Proposition 2.2.6 of [3]). Passing to the limit as m' goes to ∞ , we obtain a homomorphism $\tilde{\varphi} : A \rightarrow \mathbb{R}[[t]]$ lifting φ . It follows from Lemma 2.12 that $\tilde{\varphi}(f)$ must either be 0 or have positive leading coefficient, and we thus conclude the same for $\varphi(f)$. Thus, f is strongly nonnegative. \square

3. OBSTRUCTIONS TO SUMS OF SQUARES

We now apply the concept of strong nonnegativity to study obstructions to nonnegative functions being sums of squares. We will use the concept of degrees of functions, and consequently from this point on the choice of imbedding of $V(I)$ into affine space becomes relevant. Recall the following definition:

Definition 3.1. Fix $I \subseteq \mathbb{R}[x_1, \dots, x_n]$. For $f \in \mathbb{R}[x_1, \dots, x_n]$ and $k \geq 1$, we say that f is k -sos modulo I if there exist $g_1, \dots, g_m \in \mathbb{R}[x_1, \dots, x_n]$ of degree at most k such that

$$f \equiv \sum_{i=1}^m g_i^2 \pmod{I}.$$

Given $d, k \geq 1$, we say that I is (d, k) -**sos** if every $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree at most d which is nonnegative on $V_{\mathbb{R}}(I)$ is k -sos modulo I .

Note that if f is k -sos modulo I , then f is nonnegative on $V_{\mathbb{R}}(I)$, so the latter definition says that as many functions as possible (of degree at most d) are k -sos modulo I .

Proposition 2.4 (3) then trivially implies:

Corollary 3.2. *Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. If there exists a function $f \in A$ of degree less than or equal to d which is nonnegative on $V_{\mathbb{R}}(I)$ but not strongly nonnegative, then I is not (d, k) -sos for any k .*

We now specialize to linear functions, and recover an obstruction theorem of Gouveia and Netzer; see Theorem 4.5 of [4]. To give the statement, we define:

Definition 3.3. A point $P \in V_{\mathbb{R}}(I)$ is **convex-singular** if it is a singular point of $V(I)$, it lies on the relative boundary of $\text{conv}(V_{\mathbb{R}}(I))$, and the tangent space to $V(I)$ at P meets the relative interior of $\text{conv}(V_{\mathbb{R}}(I))$.

Remark 3.4. Note that the tangent space of $V(I)$ at P is canonically a subspace of the tangent space at P of the ambient affine space \mathbb{R}^n , which is canonically identified via translation with \mathbb{R}^n itself. Thus the definition makes sense.

Remark 3.5. Our definition differs slightly from that of [4], which considers instead the tangent space of $V(\sqrt[n]{I})$, where $\sqrt[n]{I}$ is the real radical ideal associated to I . For instance, the origin in \mathbb{R}^3 is convex-singular in $V(x^2 + y^2)$ in our definition, but not in [4]. Indeed, we consider a point to be its own relative interior, so for us the origin in \mathbb{R}^2 is also convex-singular in $V(x^2 + y^2)$.

The obstruction theorem is then the following:

Theorem 3.6. *Suppose we have $I \subseteq \mathbb{R}[x_1, \dots, x_n]$, and $P \in V_{\mathbb{R}}(I)$ is convex-singular. Then I is not $(1, k)$ -sos for any k .*

Proof. We claim that there is a linear function f which is nonnegative on $V_{\mathbb{R}}(I)$, vanishes at P , and induces a nonzero linear function on the tangent space of $V(I)$ at P . In the case that $V_{\mathbb{R}}(I) = \{P\}$, this is trivial: we may take any f whose zero set contains P but not the tangent space at P . Thus suppose $V_{\mathbb{R}}(I)$ is not a single point. If we choose a sequence of points in the affine hull of $V_{\mathbb{R}}(I)$ but outside $\text{conv}(V_{\mathbb{R}}(I))$ converging to P , the Separation Theorem (Theorem III.1.3 in [2]) gives us a sequence of linear functions on the affine hull, nonnegative on $V_{\mathbb{R}}(I)$ and negative on the points in our sequence. Taking a suitable limit of these (rescaling as necessary) gives a nonzero linear function \tilde{f} on the affine hull, nonnegative on $V_{\mathbb{R}}(I)$, and with $\tilde{f}(P) = 0$. We then have that \tilde{f} must be strictly positive on the relative interior of $V_{\mathbb{R}}(I)$. Choose f to be any lift of \tilde{f} to a linear function on \mathbb{R}^n . Now, since f is linear it induces the same function on the tangent space to \mathbb{R}^n at P , and via restriction on the tangent space to $V(I)$ at P . By hypothesis the latter tangent space meets the relative interior of $V_{\mathbb{R}}(I)$, so we see that the induced function on the tangent space is nonzero, completing the proof of the claim.

Now, because f induces a nonzero linear function on the tangent space, there is a tangent vector on which f is negative, and this corresponds to a homomorphism $\varphi : \mathbb{R}[x_1, \dots, x_n]/I \rightarrow \mathbb{R}[\epsilon]/(\epsilon^2)$ at P sending f to a negative multiple of ϵ . Thus, f is not strongly nonnegative. By Corollary 4.3, we have that f is not a sum of squares, and hence I is not $(1, k)$ -sos for any k . \square

Hypersurfaces present a particularly nice case of the theorem.

Corollary 3.7. *Suppose $I = (g)$ is principal in $\mathbb{R}[x_1, \dots, x_n]$, and suppose $P \in V_{\mathbb{R}}(I)$ is a singularity lying on the boundary of $\text{conv}(V_{\mathbb{R}}(I))$. Then I is not $(1, k)$ -sos for any k .*

Proof. The variety $V(I)$ has codimension one, so the tangent space at the singular point 0 is all of \mathbb{R}^n . Thus, P is convex-singular, and we conclude the desired result from Theorem 3.6. \square

The following example is a basic example of applying the theorem on convex singularities.

Example 3.8. Consider the ideal $I = (y^2 - x^3) \subseteq \mathbb{R}[x, y]$ from Example 2.6. The singular point $P = (0, 0)$ of $V(I)$ lies on the boundary of $\text{conv}(V_{\mathbb{R}}(I))$, so by Corollary 3.7 we have that I is not $(1, k)$ -sos for any k . Of course, this also follows from Corollary 3.2 and Example 2.6. As in the earlier example, this may be made compact by instead setting $I = (y^2 - x^3 + x^4)$.

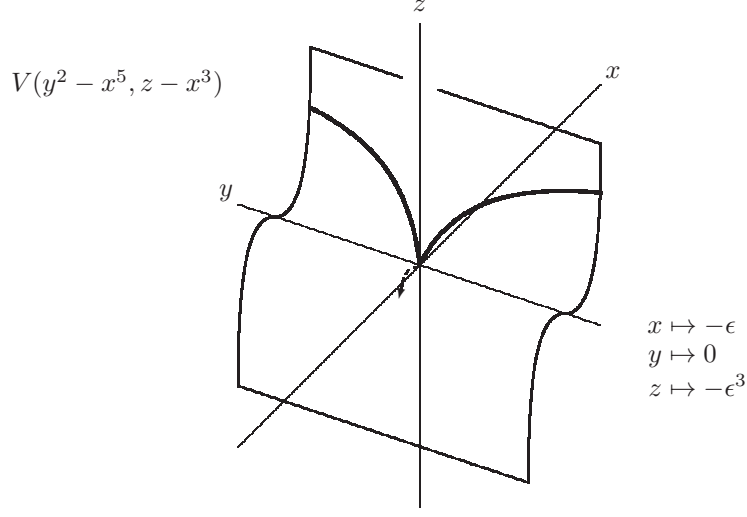


FIGURE 4. A higher-order infinitesimal arc on $V(y^2 - x^5, z - x^3)$ pointing in the negative direction.

However, we also see that Corollary 3.2 works more generally than for convex singularities. Indeed, convex singularities may be viewed as causing strong non-negativity to fail at first order, while the general definition requires examining all orders.

Example 3.9. Consider the ideal $I = (y^2 - x^5, z - x^3) \subseteq \mathbb{R}[x, y, z]$, and the function $f(x, y, z) = z$ nonnegative on $V_{\mathbb{R}}(I)$. The only singular point of $V(I)$ is $P = (0, 0, 0)$, and the tangent space to $V(I)$ at P is precisely the plane $z = 0$, so P is not a convex singularity. However, $V(I)$ has higher-order infinitesimal arcs pointing into the negative direction of z , for instance given by the homomorphism

$$\varphi : \mathbb{R}[x, y, z]/(y^2 - x^5, z - x^3) \rightarrow \mathbb{R}[\epsilon]/(\epsilon^4), \quad \varphi(x) = -\epsilon, \quad \varphi(y) = 0, \quad \varphi(z) = -\epsilon^3$$

at P . Once again, we see that f is not strongly nonnegative, and we conclude by Corollary 3.2 that I is not TH_k -exact for any k .

This example may also be made compact, by setting $I = (y^2 - x^5 + x^6, z - x^3)$.

However, we see that strong nonnegativity still has limitations in its ability to recognize functions which are not sums of squares. For instance, in Example 2.6, if we took $f = x + c$ for any $c > 0$ we would have a function which is strictly positive, and hence strongly nonnegative, but still not a sum of squares modulo I . However, Schmüdgen's Positivstellensatz implies (see Corollary 3 of [9]) that if $V_{\mathbb{R}}(I)$ is compact and f is strictly positive, then f is a sum of squares. Since strong nonnegativity lies between nonnegativity and strict positivity, it is natural to wonder if a strongly nonnegative function is a sum of squares when $V_{\mathbb{R}}(I)$ is compact. The following example shows that this is not the case.

Example 3.10. Let $I = (x_1^2 + \cdots + x_n^2 - 1)$ be the ideal of the sphere in \mathbb{R}^n , with $n \geq 4$. According to Theorem 2.6.3 of [8], there exists a polynomial function f which is nonnegative on $V_{\mathbb{R}}(I)$ but not a sum of squares modulo I . Since $V(I)$ is nonsingular, we have by Theorem 2.10 that f is strongly nonnegative on $V(I)$.

If we wish to have an example with f linear, we may simply add an additional variable y , and add to I the relation $y = f$, so that the resulting coordinate rings are isomorphic. Then y is strongly nonnegative, but is not a sum of squares modulo I .

4. OBSTRUCTIONS TO THETA EXACTNESS

Recall that the closure of the convex hull of a real variety $V_{\mathbb{R}}(I)$ can be described as the intersection of all halfspaces defined by linear functions nonnegative on it. Determining a description of the closure of the convex hull of a real variety in terms of finitely many polynomial equations and inequalities is difficult in general. To combat this, Gouveia, Parrilo and Thomas [5] introduce a hierarchy of nested spectrahedral shadows containing the convex hull of $V_{\mathbb{R}}(I)$. The k -th theta body denoted $TH_k(I)$ is precisely

$$TH_k(I) = \{x \in \mathbb{R}^n : f(x) \geq 0 \ \forall f \text{ linear and } k\text{-sos mod } I\}$$

These theta bodies form a hierarchy of relaxations

$$TH_1(I) \supseteq TH_2(I) \supseteq \cdots \supseteq \overline{\text{conv}(V_{\mathbb{R}}(I))}$$

of the closure of the convex hull of $V_{\mathbb{R}}(I)$. When the k -th theta body coincides with $\overline{\text{conv}(V_{\mathbb{R}}(I))}$, I is said to be **TH_k -exact**. These two concepts are related by the following proposition from [5]; see Proposition 5.2 below for a stronger statement.

Proposition 4.1. *Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. If I is $(1, k)$ -sos then I is TH_k -exact.*

Moreover, Gouveia, Parrilo and Thomas also proved the following remarkable converse. See Corollary 2.12 of [5].

Theorem 4.2. *Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be a real radical ideal. Then I is $(1, k)$ -sos if and only if I is TH_k -exact.*

This converse theorem, together with our results on obstructions to an ideal being $(1, k)$ -sos, immediately allow us to rephrase the latter results in the real radical case in terms of obstructions to theta exactness. We thus conclude:

Corollary 4.3. *Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be a real radical ideal. If there exists a linear function f that is nonnegative on $V_{\mathbb{R}}(I)$ but not strongly nonnegative, then I is not TH_k -exact for any k . \square*

The obstruction theorem of Gouveia and Netzer as they stated it is equivalent to the following:

Theorem 4.4. *Suppose we have $I \subseteq \mathbb{R}[x_1, \dots, x_n]$, and $P \in V_{\mathbb{R}}(I)$ is a convex-singular point of $V(\sqrt[k]{I})$, where $\sqrt[k]{I}$ is the real radical ideal associated to I . Then I is not TH_k -exact for any k .*

Proof. We conclude from Theorem 3.6 that $\sqrt[k]{I}$ is not $(1, k)$ -sos, and thus Theorem 4.2 implies that $\sqrt[k]{I}$ is not TH_k -exact. Since $TH_k(\sqrt[k]{I}) \subseteq TH_k(I)$, we conclude the desired statement. \square

Similarly, we conclude:

Corollary 4.5. *Suppose $I = (g)$ is principal and real radical in $\mathbb{R}[x_1, \dots, x_n]$, and suppose $P \in V_{\mathbb{R}}(I)$ is a singularity lying on the boundary of $\text{conv}(V_{\mathbb{R}}(I))$. Then I is not TH_k -exact for any k .*

As before, Example 3.9 gives an example in which Corollary 4.3 goes further than Theorem 4.4; indeed, in this case the ideal is real radical, so we conclude that it is not TH_k -exact for any k .

5. A NEW SUM OF SQUARES CONDITION

Finally, we consider a weaker notion of $(1, k)$ -sos arising from strong nonnegativity.

Definition 5.1. Given $d, k \geq 1$, and an ideal $I \subseteq \mathbb{R}[x_1, \dots, x_n]$, we say that I is **weakly (d, k) -sos** if for every $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree at most d which is strongly nonnegative on $V_{\mathbb{R}}(I)$, we have that f is k -sos.

Though being weakly $(1, k)$ -sos relaxes the notion of being $(1, k)$ -sos, it still implies TH_k -exactness. This generalizes Lemma 1.5 of [5].

Proposition 5.2. *If I is weakly $(1, k)$ -sos, then I is TH_k -exact.*

Proof. Let $P \in \mathbb{R}^n$ such that $P \notin \overline{\text{conv}(V_{\mathbb{R}}(I))}$. By the Separation Theorem, there is a linear polynomial f such that f is nonnegative on $\text{conv}(V_{\mathbb{R}}(I))$ and $f(P) < 0$. Consider the linear function $g = f - \frac{f(P)}{2}$. We have $g(P) < 0$, and g is *positive* on $\overline{\text{conv}(V_{\mathbb{R}}(I))}$ and hence positive on $V_{\mathbb{R}}(I)$. This implies g is strongly nonnegative by Proposition 2.4 (2), and g is then a sum of squares of polynomials of degree at most k by hypothesis. Since P was arbitrary outside $\overline{\text{conv}(V_{\mathbb{R}}(I))}$, the result follows. \square

The above proposition along with Theorem 4.2 shows that for real radical ideals, being weakly $(1, k)$ -sos is in fact equivalent to being TH_k -exact.

Corollary 5.3. *If $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ is a real radical ideal, then the following are equivalent:*

- (1) I is weakly $(1, k)$ -sos
- (2) I is $(1, k)$ -sos
- (3) I is TH_k -exact.

Proof. Proposition 5.2 establishes that (1) \implies (3). (3) \implies (2) follows from Theorem 4.2. Finally, (2) \implies (1) is immediate from Proposition 2.4 (1). \square

We conclude by briefly discussing some further questions. Our original hope was that replacing $(1, k)$ -sos with weakly $(1, k)$ -sos would allow the relaxation of the radical portion of the real radical condition in Theorem 4.2. We have not yet obtained any results in this direction, but neither do we have any counterexamples. Indeed, we are not aware of any examples of an ideal I which is TH_k -exact but not weakly $(1, k)$ -sos. It seems implausible that the two conditions should be equivalent without any sort of hypothesis implying at least that $V_{\mathbb{R}}(I)$ is Zariski dense in $V_{\mathbb{C}}(I)$, but neither is it entirely absurd: we note that if $V_{\mathbb{C}}(I)$ is irreducible and $V_{\mathbb{R}}(I)$ is not Zariski dense, then we will have that $V_{\mathbb{R}}(I)$ is contained in the singular locus of $V(I)$. In addition, if $V_{\mathbb{R}}(I) = \emptyset$, then according to the Positivstellensatz (2.2.1 of [8]) we have -1 a sum of squares modulo I , which then implies that every polynomial is a sum of squares modulo I .

It would also be interesting to consider effectiveness questions. Of course, the concept of strong nonnegativity is already useful from an effectiveness point of view insofar as it provides a new approach to producing a certificate that a given function is not a sum of squares modulo an ideal. However, it is also natural to

wonder whether, for instance, it is possible to effectively determine whether a given function is strongly nonnegative at a point. This question naturally breaks up into two subquestions: whether strong nonnegativity at a point can be effectively computed for a given order of infinitesimal arcs, and whether for any given ideal I , point $P \in V_{\mathbb{R}}(I)$, and function f , one can effectively compute a number N such that to determine that f is strongly nonnegative at P , it is enough to look at infinitesimal arcs of order up to N . The latter question is interesting in and of itself, both from a theoretical point of view, and because one could envision that even if the answer to the first question is negative in general, it could be positive in some more specific scenarios.

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